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LETTER TO THE EDITOR

On the analytic structure of the driven pendulum

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Abstract. The analytic structure of the solution of the driven pendulum is investigated through Painlevé analysis in the complex time plane. The existence is pointed out of a two-armed infinite sheeted Riemann structure of the singularities after an exponential transformation.

We consider the general form of the equation of motion of the driven pendulum [1], given by

$$\ddot{x}+\alpha\dot{x}+\omega_0^2\sin x=\gamma\cos\omega t \qquad \cdot = \mathrm{d}/\mathrm{d}t \tag{1}$$

where ω_0^2 is the natural frequency of the pendulum, α is the viscous damping parameter, γ and ω are, respectively, the amplitude and frequency of the external periodic force. Here we wish to investigate the non-integrability aspects of the system (1) by studying the nature of the singularities exhibited by the solution in the complex time plane.

It is well known that the Painlevé (P-) analysis [2–5] can be profitably used not only to investigate the integrability aspects [3–5] of dynamical systems, but also to analyse the non-integrability aspects, especially through the analytic structure studies [6–12] of the solution of the equation of motion. Most of the dynamical systems which have been studied recently for their analytic structure in the non-integrable case are of polynomial type such as the coupled anharmonic oscillators [4, 5], the Henon–Heiles system [6], the Lorenz system [8], the Duffing oscillator [7–12] and so on. However, very few dynamical systems have been studied in this way which have their equations of motion with non-polynomial type such as the Toda lattice [6], the sine-Gordon equation and so on. In this letter we present the analytic structure of the driven pendulum (1) and show that the singularities exhibit a complicated, clustered, two-armed multisheeted Riemann structure in the complex t-plane, after making an exponential transformation.

Introducing the variables:

$$y=\mathrm{e}^{\mathrm{i}x} \qquad \text{and} \qquad \tilde{t}=-\mathrm{i}t \tag{2}$$

(1) reduces (after dropping the tilde) to

$$y\ddot{y}-\dot{y}^2+\mathrm{i}\alpha y\dot{y}+\tfrac{1}{2}\omega_0^2 y-\tfrac{1}{2}\omega_0^2 y^3+\mathrm{i}\gamma y^2\cosh\omega t=0$$
$$\cdot=\mathrm{d}/\mathrm{d}t. \tag{3}$$

We will analyse the singularity structure of the solution to this equation. The general solution to (3) can be represented locally as a Laurent series of the form

$$y=\sum_{j=0}^{\infty}a_j\tau^{j-2} \qquad \tau=(t-t_0)\to 0 \tag{4}$$

about an arbitrary movable singularity t_0, in which one of the a_js must be arbitrary in addition to t_0. Direct substitution of the ansatz (4) into (3) yields the recursion relations for the a_js:

$$\sum_r \left(a_{j-r}a_r(j-r-2)(j-2r-1)+\mathrm{i}\alpha a_{j-r-1}a_r(j-r-3)-\tfrac{1}{2}\omega_0^2\sum_p a_{j-r}a_{r-p}a_p \right.$$
$$\left. +\mathrm{i}\gamma\sum_p G_{j-r-2}a_{r-p}a_p\right)$$
$$=-\tfrac{1}{2}\omega_0^2 a_{j-4}, \qquad 0\leqslant p\leqslant r\leqslant j \tag{5}$$

where

$$G(t)=\cosh\omega t \text{ and } G_n=\frac{1}{n!}\left.\frac{\partial^n G(t)}{\partial t^n}\right|_{t=t_0}.$$

From (5) one obtains

$$j=0 \qquad a_0=4/\omega_0^2 \tag{6a}$$
$$j=1 \qquad a_1=-\mathrm{i}4\alpha/\omega_0^2 \tag{6b}$$
$$j=2 \qquad 0\cdot a_2+(2\alpha^2+\mathrm{i}\gamma\cosh\omega t_0)a_0^2=0. \tag{6c}$$

Equation (6*c*) gives the compatability condition that ensures the arbitrariness of a_2. This will be satisfied if, and only if, both α and γ become zero for arbitrary t_0. Thus (3) is of *P*-type only when both $\alpha=0$ and $\gamma=0$, in which case the system obviously become integrable in terms of Jacobian elliptic functions.

If $\alpha\neq 0$ and $\gamma\neq 0$, the arbitrariness of a_2 can be recaptured by modifying the ansatz (4) and introducing logarithmic terms in (4) through the psi series [7]

$$y=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}\tau^{j-2}(\tau^2\ln\tau)^k. \tag{7}$$

Then the recursion relation for the a_{jk}s for (3) becomes

$$\sum_{r,s}\left(a_{j-r,k-s}a_{rs}(j-r+2k-2s-2)(j-2r+2k-4s-1)\right.$$
$$+a_{j-r-2,k-s+1}a_{rs}[(2j-2r+4k-4s-5)(k-2s+1)-s]$$
$$+a_{j-r-4,k-s+2}a_{rs}(k-s+2)(k-2s+1)$$
$$+\mathrm{i}\alpha a_{j-r-1,k-s}a_{rs}(j-r+2k-2s-3)+\mathrm{i}\alpha a_{j-r-3,k-s+1}a_{rs}(k-s+1)$$
$$\left.-\tfrac{1}{2}\omega_0^2\sum_{p,q} a_{j-r,k-s}a_{r-p,s-q}a_{pq}+\mathrm{i}\gamma\sum_p G_{j-r-2}a_{r-p,k-s}a_{ps}\right)$$
$$=-\tfrac{1}{2}\omega_0^2 a_{j-4,k} \qquad 0\leqslant p\leqslant r\leqslant j \quad 0\leqslant q\leqslant s\leqslant k. \tag{8}$$

The values of the coefficients a_{00} and a_{10} are given by $a_{00}=4/\omega_0^2$ and $a_{10}=-\mathrm{i}4\alpha/\omega_0^2$. For a_{20} to be arbitrary we now have

$$0\cdot a_{20}-a_{01}a_{00}+(2\alpha^2+\mathrm{i}\gamma\cosh\omega t_0)a_{00}^2=0 \tag{9}$$

which means that

$$a_{01}=4(2\alpha^2+\mathrm{i}\gamma\cosh\omega t_0)/\omega_0^2. \tag{10}$$

From (7) we see that the singularity t_0 is no longer a movable pole but is, instead, a movable logarithmic branch point and (3) is not of P-type. Thus the system (1) is, in general, non-integrable except when both $\alpha = 0$ and $\gamma = 0$.

In order to study the analytic structure of the solution of (3) we now look for a closed set of recursion relations amongst the a_{jk}s. These turn out to be the set a_{0k} $k = 0, 1, 2, \ldots$, which satisfy

$$\sum_s \Big([8(k-s)(k-s-1) - 8s(k-s) + 8s - 4(k-s) + 4] \times a_{0,k-s}a_{0s} - \omega_0^2 \sum_q a_{0,k-s}a_{0,s-q}a_{0q} \Big) = 0. \tag{11}$$

Introducing the generating function

$$\Theta(z) = \sum_{k=0}^{\infty} a_{0k}z^k \tag{12}$$

where z is a function of τ, the following differential equation for $\Theta(z)$ is obtained:

$$8z^2\Theta\Theta'' - 8z^2\Theta'^2 + 4z\Theta\Theta' + 4\Theta^2 - \omega_0^2\Theta^3 = 0 \tag{13}$$

where prime denotes differentiation with respect to z. Since in the limit $\tau \to 0$, the most dominant terms in the psi series (7) involve powers of $\tau^2 \ln \tau$ only, we can obtain (13) in a more direct way by substituting

$$y = \frac{1}{\tau^2}\Theta(z) \tag{14}$$

where

$$z = \tau^2 \ln \tau \tag{15}$$

into (3). Thus (13) can be regarded as the original (3) rescaled in the neighbourhood of a given singularity t_0. Furthermore, it is a straightforward exercise to show that (13) has the Painlevé property with $\Theta(z)$ having local expansion

$$\Theta(z) = \sum_{j=0}^{\infty} A_j(z - z_0)^{j-2} \tag{16}$$

in which A_2 and z_0 are the arbitrary parameters.

We can also see that (13) can be integrated exactly by making the substitution

$$\Theta(z) = \xi^2 f(\xi) \qquad \xi = \sqrt{z} \tag{17}$$

in (13) so that we get

$$ff'' - f'^2 - \tfrac{1}{2}\omega_0^2 f^3 = 0 \tag{18}$$

where prime refers to differentiation with respect to ξ. The first integral of (18) is given by [13]

$$f'^2 = \omega_0^2 f^3 + I_1 f^2 \tag{19}$$

where the value of the integration constant can be defined as $I_1 = -3\omega_0^2 a_{01} = -12(2\alpha^2 + \mathrm{i}\gamma \cosh \omega t_0)$. By a simple transformation

$$f(\xi) = [g^2(\xi) - 1]I_1/\omega_0^2 \tag{20}$$

(19) is reduced to a simple first-order nonlinear ordinary differential equation:

$$g'(\xi)=\tfrac{1}{2}\sqrt{I_1}\,[1-g^2(\xi)]. \tag{21}$$

Equation (21) can be readily integrated and its solution is given by

$$g(\xi)=\tanh[\tfrac{1}{2}\sqrt{I_1}\,(\xi-\xi_0)] \tag{22}$$

where ξ_0 is the arbitrary integration constant. Choosing $\xi_0=0$ for convenience, we can write

$$f(\xi)=-(I_1/\omega_0^2)\,\mathrm{sech}^2(\tfrac{1}{2}\sqrt{I_1}\xi). \tag{23}$$

It is evident that $f(\xi)$ has poles of second order which are situated at the discrete points

$$\xi_m=\mathrm{i}\frac{\pi}{\sqrt{I_1}}(2m+1) \qquad m\in\mathbb{Z} \tag{24}$$

in the complex ξ plane, where m denotes the lattice site integer.

The singularity positions in the z plane can be obtained from (cf (17)) the pole positions of ξ_m as

$$z_m=\tfrac{1}{12}\pi^2(2m+1)^2\frac{(2\alpha^2-\mathrm{i}\gamma\cosh\omega t_0)}{(4\alpha^4+\gamma^2\cosh^2\omega t_0)}. \tag{25}$$

From (25) we can study the singularity structure in the complex z plane by plotting z_{Im} versus z_{Re} for a chosen set of parametric values. As an illustration, in figure 1 we have fixed the parameter values as $\alpha=0.3$, $\omega_0^2=1.0$, $\omega=0.5$ and obtained the singularity structure in the complex z-plane about the singularity located at the origin (since we take $t_0=0$) for $\gamma=0.5$. This singularity pattern, given by (25), can be mapped back to the complex t-plane by the multivalued transformation (cf (15) where we have chosen $t_0=0$)

$$z=t^2\ln t \tag{26}$$

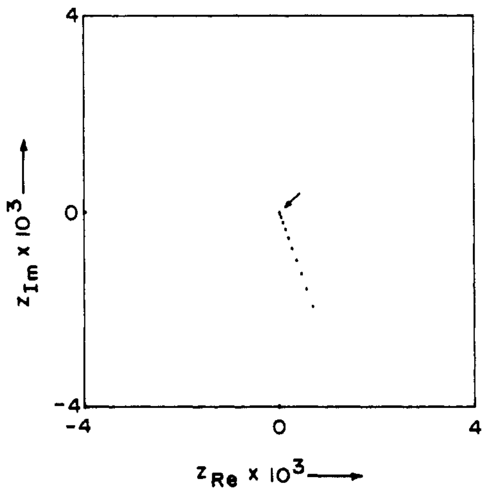

Figure 1. Singularity structure in the complex z-plane ($z=t^2\ln t$), given by (25), for $\alpha=0.3$, $\omega_0^2=1.0$, $\omega=0.5$, $m=-10,-9,\ldots,-1,0,1,\ldots,9,10$ and $\gamma=0.5$.

similar to the procedure adopted by Fournier, Levine and Tabor [7] for the Duffing oscillator. This can be performed by using polar coordinates in both z- and t-planes as

$$z = \rho\, e^{i\phi} \qquad \text{and} \qquad t = r\, e^{i\theta}. \tag{27}$$

From (26) and (27) we can write the real and imaginary parts of z in terms of r and θ as

$$\operatorname{Re} z = r^2[\cos(2\theta)\ln r - (\theta + 2\pi n)\sin(2\theta)] \tag{28a}$$

$$\operatorname{Im} z = r^2[\sin(2\theta)\ln r + (\theta + 2\pi n)\cos(2\theta)] \tag{28b}$$

where n is the Riemann sheet index in the t-plane. From (28), it follows that

$$r = \exp[-(\theta + 2\pi n)\cot(2\theta - \phi) \tag{29}$$

and so

$$\rho = -(\theta + 2\pi n)\operatorname{cosec}(2\theta - \phi)\exp[-2(\theta + 2\pi n)\cot(2\theta - \phi)]. \tag{30}$$

Equations (29) and (30) completely determine the mapping $z \to t$.

For a given pole in the z-plane given by (25), we assign polar coordinates ρ and ϕ which can be readily computed. Then from (30) we can compute the value of θ, by a simple numerical root search method, for any sheet n, corresponding to the given (ρ, ϕ) values. From this value of θ the associated r value is computed from (29). Thus for any one of the singularities in the z-plane given by (25), we can obtain the corresponding singularity and its substructure through the analytic mapping (29) and (30). In figure 2 we have shown one such local singularity structure in the complex t-plane, in the neighbourhood of the marked singularity in figure 1, determined from the analytic mapping for the same choice of parameteric values as mentioned above. From figure 2 we find that the local singularity structure obtained is a two armed structure with the singularities becoming densely 'packed' and clustered along each arm. As they approach the centre of the two arms, with n increasing. The recursive nature of this clustering leads to an immensely complicated singularity structure in

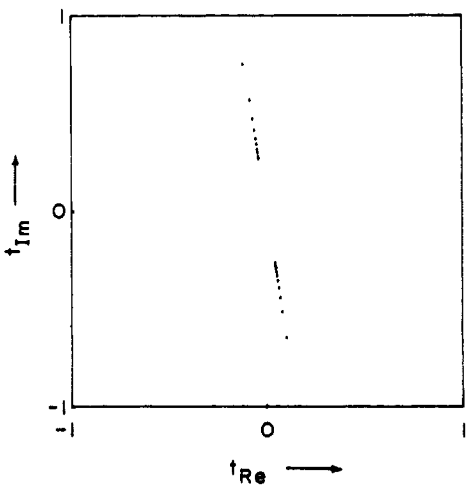

Figure 2. Local singularity structure in the complex t-plane in the neighbourhood of the marked singularity in figure 1, determined from the analytic mapping (29) and (30) for $\alpha = 0.3$, $\omega_0^2 = 1.0$, $\omega = 0.5$, $m = 0$ and $\gamma = 0.5$.

the complex t-plane. This can be further checked by directly integrating the equation of motion (1) in the complex t-plane using the ATOMFT integrator, developed by Chang [14], thereby obtaining the singularity structure in the complex t-plane. Work along these lines is in progress and will be reported elsewhere.

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